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Total Linearization of Probability Density Evolution Equations

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Total linearization of the Boltzmann and Vlasov equations is used to present a technique applicable to equations which have polynomial-type nonlinearities and describe the evolution of probability density functions or distributions. It is an extension of an earlier result, where a method yielding a linear equation with a nonlinear constraint was presented.

Let $f \in X$ and $g \in X \otimes X$, where X is a space containing all evolutions of probability density functions and distributions on $\mathbb{R}^6$, e.g., $X = \mathscr{D}'(\mathbb{R}^7)$.

Let

$$Y = Y_0 \oplus Y_1 \oplus Y_2 \oplus Y_4 \oplus \cdots \oplus Y_{2^k} \oplus \cdots$$

where $Y_j = X_1 \otimes X_2 \otimes \cdots \otimes X_j$, and each X_i is a copy of X. Total linearization of kinetic equations is achieved in the space Y. The Boltzmann equation [adapted from Percus (1987)] can be written as

$$\begin{aligned}(\partial f/\partial t)(t, x, v_1) + v_1 \cdot [(\nabla_x f)(t, x, v_1)] + 1/m\, F(x) \cdot [(\nabla_v f)(t, x, v_1)] \\ = \int_{\mathbb{R}^3} \int_{\Omega} |v_1 - v_2| \sigma(|v_1 - v_2|, \Omega)[f(t, x, v_1')f(t, x, v_2') \\ - f(t, x, v_1)f(t, x, v_2)]\, d\Omega\, d^3 v_2 \qquad (1)\end{aligned}$$

where σ is the differential cross section, Ω is the direction of scattering, $F(\cdot)$ is an external force field, and v_1 and v_2 are precollisional velocities,

$$v_1'(v_1, v_2, \Omega) = (v_1 + v_2 + |v_1 - v_2|\Omega)/2$$

$$v_2'(v_1, v_2, \Omega) = (v_1 + v_2 - |v_1 - v_2|\Omega)/2$$

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Define Π_6 by

$$(\Pi_6 g)(t, x, v) = \int\int_{\mathbb{R}^6} g(t, x, v, 0, y, u)\, d^3y\, d^3u$$

Since we are dealing with the evolution of probability density functions, zero in the integrand can be replaced by any nonnegative number s. We can easily see that Π_6 is linear.

Also, since for all $t > 0$, $f(t)$ is a nonnegative function on $\mathbb{R}^6$ and belongs to the unit sphere in $L^1(\mathbb{R}^6)$, $\Pi_6(f \otimes f) = f$, i.e., Π_6 coincides with the square root in the sense of tensor product.

Defining

$$(A_0 g)(t, x, v) = (Lg)(t, x, v) - v \cdot [\nabla_x(\Pi_6 g)](t, x, v) - 1/m\, F(x) \cdot [\nabla_v(\Pi_6 g)](t, x, v)$$

$$(Lg)(t, x, v_1) = \int_{\mathbb{R}^3} \int_{\Omega} |v_1 - v_2| \sigma(|v_1 - v_2|, \Omega)[g(t, x, v_1', t, x, v_2') - g(t, x, v_1, t, x, v_2)]\, d\Omega\, d^3v_2$$

we obtain the following linear form of the Boltzmann equation (Grzybowski, 1988)

$$\partial/\partial t\, (\Pi_6 g) = A_0 g \tag{2}$$

with the nonlinear constraint

$$g = (\Pi_6 g) \otimes (\Pi_6 g) \tag{3}$$

This nonlinearity can be removed by redefining the constraint in spaces Y_{2^k} for higher and higher values of k and considering the limiting case $k \to \infty$. Define

$$A_1 = \Pi_6 \otimes \Pi_6, \qquad A_2 = \Pi_{12} \otimes \Pi_{12},$$
$$A_3 = \Pi_{24} \otimes \Pi_{24}, \ldots, \qquad A_k = \Pi_{3*2^k} \otimes \Pi_{3*2^k}, \ldots$$

where

$$(\Pi_{3*2^k} h)(t_1, x_1, v_1, t_2, x_2, v_2, t_3, x_3, v_3, \ldots, t_{2^k}, x_{2^k}, v_{2^k}) = \int \cdots \int_{\mathbb{R}^{3*2^k}} h(t_1, x_1, v_1, 0, x_2, v_2, 0, x_3, v_3, \ldots, 0, x_{2^k}, v_{2^k}) \times d^3x_1\, d^3v_1 \cdots d^3x_{3*2^k}\, d^3v_{3*2^k}$$

for $h = f_1 \otimes f_2 \otimes f_3 \otimes \cdots \otimes f_{2^k}, f_i \in X$.

By linear extension, operators A_i can be defined on respective tensor product spaces and we can see that

$$A_1((f_1\otimes f_2)\otimes(f_3\otimes f_4))=f_1\otimes f_3,$$

$$A_2((f_1\otimes f_2)\otimes(f_3\otimes f_4)\otimes(f_5\otimes f_6)\otimes(f_7\otimes f_8))$$
$$=(f_1\otimes f_2)\otimes(f_5\otimes f_6),$$

etc., so that each A_k acts like a tensor square root on certain elements of $Y_{2^{k+1}}$, for example, on tensor powers of order 2^{k+1}.

Define also

$$A_L=\begin{bmatrix} A_0 & 0 & 0 & \cdots \\ 0 & A_1 & 0 & \cdots \\ 0 & 0 & A_2 & \cdots \\ \cdots \end{bmatrix}, \qquad B_L=\begin{bmatrix} \partial/\partial t\,\Pi_6 & 0 & 0 & \cdots \\ I & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ \cdots \end{bmatrix}$$

$$G_L=\begin{bmatrix} g^{(1)} \\ g^{(2)} \\ g^{(4)} \\ g^{(8)} \\ \vdots \end{bmatrix}$$

where $g^{(k)}\in Y_k$. Then (2) and (3) give

$$B_L G_L=A_L G_L \tag{4}$$

which is a linear equation with no constraints.

Defining

$$(A_{\#}g)(t,x,v)=(L_{\#}g)(t,x,v)-v\cdot[\nabla_x(\Pi_6 g)](t,x,v)$$
$$-1/m\,F_1(x)\cdot[\nabla_v(\Pi_6 g)](t,x,v)$$

$$(L_{\#}g)(t,x,v)=-1/m\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}F_2(x-x')\cdot\nabla_v g(t,x,v,t,x',v')\,d^3x'\,d^3v'$$

we obtain the following linear form of the Vlasov equation:

$$\partial/\partial t(\Pi_6 g)=A_{\#}g. \tag{5}$$

with the nonlinear constraint (3).

Consequently, if we define $A_{\#L}$ by replacing A_0 with $A_{\#}$

$$A_{\#L}=\begin{bmatrix} A_{\#} & 0 & 0 & \cdots \\ 0 & A_1 & 0 & \cdots \\ 0 & 0 & A_2 & \cdots \\ \cdots \end{bmatrix}$$

then (5) and (3) give

$$B_L G_L = A_{\# L} G_L \tag{6}$$

which is a linear form of the Vlasov equation with no constraints.

It is not necessary to use full tensor products $f \otimes f$, $f \otimes f \otimes f \otimes f \otimes f$, etc. Defining $g(t, x, v, y, u) = f(t, x, v) \cdot f(t, y, u)$ (and properly redefining related objects) suffices, but it can be done at the price of imposing stronger assumptions on f, if we want the (conditionally linear) tensor square root operation to commute with the action of $\partial/\partial t$.

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